Einstein–Proca model: spherically symmetric solutions

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Abstract

The Proca wave equation describes a classical massive spin 1 particle. We analyze the gravitational interaction of this vector field. In particular, the spherically symmetric solutions of the Einstein-Proca coupled system are obtained numerically. Although at infinity the metric field approaches the usual Schwarzschild (Reissner-Nordström) limit, we demonstrate the absence of black hole type configurations.

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I. INTRODUCTION

A massive vector meson (spin 1 particle with a non-trivial mass) is described by a one-form field which obeys the Proca wave equation [1]. Early development of the Proca theory was concerned with the classical and quantum electrodynamics of a massive photon. However, strong experimental limits on photon's mass (see, e.g., [2]) in combination with theoretical arguments based on the idea of gauge invariance (which ultimately led to the standard model of electroweak interactions) have closed the electrodynamical chapter in the history of this theory. A further discussion of the differences between the Proca and electromagnetic fields can be found in [3].

At present, interest in the Proca field is twofold. Firstly, the Proca model presents a convenient theoretical "laboratory" for the study of Lagrangian and Hamiltonian theories with second class constraints [4]. Secondly, although it is irrelevant for the electrodynamics, a massive vector meson often appears in the spectra of many non-trivial field theoretical models, including some classes of generalized theories of gravity. In connection with this, it is interesting to investigate the specific physical effects arising in such models due to the interaction of Proca particles with electromagnetic, gravitational and other physical and geometrical fields.

The interaction of spin 1 field with electromagnetic field is known to be free of algebraic inconsistencies as well as of acausal wave propagation ($v > c$) when the coupling is minimal (or modified by the addition of an anomalous magnetic dipole moment). However, acausal propagation anomalies arise for more general interaction Lagrangians [5]. Similarly, acausal propagation takes place (along with algebraic inconsistencies) for a Proca field coupled minimally to external torsion field [6]. Different aspects of the interaction of classical and quantum vector field with torsion have been analyzed recently in [7].

Early studies of the gravitational interaction of the Proca field were centered around the black hole issue. Qualitative analysis of the self-consistent Einstein-Proca system revealed the absence of black hole type solutions (possessing a regular horizon) with an external vec-

tor meson "hair" [8–10]. At the same time several exact spherically symmetric solutions of the Proca wave equation on the classical Schwarzschild background spacetime were obtained [11–13]. Assuming that a massive vector field source was located on a thin *spherical shell* outside the Schwarzschild horizon, it was demonstrated in [11] that the meson field may change the structure of the spacetime near the central singularity. For *point* vector field sources located at the origin [12], or at a finite distance from the origin [13], it was shown that the range of the meson field is reduced by the metric gravitational field. The energy-momentum invariant was found to be divergent on the Schwarzschild horizon. However, it should be noted that, contrary to the Abelian Proca case, the non-Abelian massive vector field (with mass of a Yang-Mills field coming from a spontaneous symmetry breaking mechanism) may form a black hole type configuration [14]. The results of numerical analysis of the spherically symmetric gravitationally interacting *complex* spin 1 field have been reported recently in [15]. In this case the Einstein-Proca system admits everywhere regular "boson star"-type solutions (cf. with massive scalar boson stars [16,17]).

A direct motivation for our current study comes from the metric-affine theory of gravity (MAG). In Einstein's general relativity the spacetime geometry is described by the curvature 2-form $R^{\alpha\beta}$. In MAG two post-Riemannian structures are introduced: the 1-form of nonmetricity $Q_{\alpha\beta}$ and the torsion 2-form T^α. For a comprehensive review of this theory see [18]. Already the early investigations [19,20] of the models with the simplest possible MAG Lagrangians, which include only a linear Hilbert term, quadratic segmental curvature invariant, and a single trace torsion or Weyl nonmetricity square term, have shown that an *effective Einstein–Proca theory* arises naturally from the vacuum MAG field equations (cf. also [12,21]). This result was subsequently extended to a very general family of MAG Lagrangians [22–24]. In all these models the effective Proca field describes the triplet of post-Riemannian one-forms which are proportional to each other: the Weyl covector $Q := g^{\alpha\beta}Q_{\alpha\beta}/4$, the torsion trace $T := e_\alpha \rfloor T^\alpha$, and the nonmetricity one-form $\Lambda := \vartheta^\alpha e^\beta \rfloor Q_{\alpha\beta} - Q$. The mass of the effective vector particle is constructed from the coupling constants of the MAG Lagrangian. For a complete review of the known exact solutions of MAG see [25].

In this paper we study the spherically symmetric static solutions of the coupled Einstein-Proca system of field equations. A preliminary analysis of the limiting cases of this problem shows a possibility of solutions which combine the exponential "Yukawa" type behavior of the Proca potential at the origin with the asymptotically Schwarzschild solution far away from the source. We will present the corresponding solutions which have been obtained by the application of numerical integration techniques.

Our main conventions and notation are taken from [18]. In particular, the η-basis of the exterior algebra is constructed from a coframe one-form ϑ^α with the help of the Hodge duality operator: $\eta^\alpha = {}^*\vartheta^\alpha, \eta^{\alpha\beta} = {}^*(\vartheta^\alpha \wedge \vartheta^\beta), \eta^{\alpha\beta\gamma} = {}^*(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma)$. The dual frame is denoted as e_α. The Greek indices $\alpha, \beta, \ldots = 0, \ldots, 3$ label anholonomic components, and the metric signature is $(-, +, +, +)$.

II. EINSTEIN-PROCA THEORY

The Lagrangian four-form of the Einstein-Proca system reads

$$V = -\frac{1}{2\kappa} R^{\alpha\beta} \wedge \eta_{\alpha\beta} - \frac{1}{2} \left(dA \wedge {}^*dA + m^2 A \wedge {}^*A \right), \tag{2.1}$$

where κ is the gravitational constant ($\kappa = \ell^2$) and m is the rest mass of the vector field A. The corresponding field equations arise from the independent variation of the action with respect to the coframe and the Proca one-forms, and read:

$$d\,{}^*dA + m^2\,{}^*A = 0, \tag{2.2}$$

$$\frac{1}{2} R^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma} = \kappa\Sigma_\alpha. \tag{2.3}$$

Here the canonical energy-momentum three-form of the massive vector field

$$\Sigma_\alpha = \frac{1}{2} \left\{ (e_\alpha \rfloor dA) \wedge {}^*dA - (e_\alpha \rfloor {}^*dA) \wedge dA + m^2 \left[(e_\alpha \rfloor A){}^*A + (e_\alpha \rfloor {}^*A) \wedge A \right] \right\}, \tag{2.4}$$

represents the usual source of the gravitational field.

It is worthwhile to recall the relationship of (2.1)-(2.3) to the MAG theory. As we have mentioned already, the same physical system arises in MAG as an effective system

[19,20,22–25] in which the effective covector Proca field is (in the notations of our previous paper [24])

$$A = \sqrt{z_4}\, k_0\, \phi. \tag{2.5}$$

Here ϕ determines the three nontrivial post-Riemannian pieces of nonmetricity and torsion (the triplet of one-forms)

$$^{(1)}T^\alpha = {}^{(3)}T^\alpha = 0, \qquad {}^{(1)}Q_{\alpha\beta} = {}^{(2)}Q_{\alpha\beta} = 0, \tag{2.6}$$

$$Q = k_0\phi, \quad \Lambda = k_1\phi, \quad T = k_2\phi. \tag{2.7}$$

The effective mass m^2 of the vector particle and the constants k_0, k_1, k_2 are constructed from the original coupling constants of the MAG Lagrangian which contains all possible quadratic invariants of the torsion and nonmetricity (11 terms) together with the linear Hilbert type term (multiplied by the constant κ) and the Weyl segmental curvature quadratic term (multiplied by the constant z_4). See [24] and [25] for more details (note, however, that in the present paper we assume that the cosmological constant is zero).

III. SPHERICALLY SYMMETRIC STATIC CASE

In terms of the local time and space coordinates (τ, r, θ, ϕ), the general spherically symmetric ansatz for the coframe can be written as

$$\vartheta^{\hat{0}} = f\, d\tau, \quad \vartheta^{\hat{1}} = \frac{g}{f}\, dr, \quad \vartheta^{\hat{2}} = r\, d\theta, \quad \vartheta^{\hat{3}} = r\sin\theta\, d\phi. \tag{3.1}$$

The geometrical meaning of the function $g(r)$ becomes evident when one computes the volume four-form

$$\eta = \vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}} = g\, r^2\, \sin\theta d\tau \wedge dr \wedge d\theta \wedge d\phi. \tag{3.2}$$

Thus, $g(r)$ measures the deviation of η from the standard spherically symmetric spacetime volume form. In a regular oriented spacetime domain we naturally have to assume

$$0 < g(r) < \infty. \tag{3.3}$$

The general static spherically symmetric configuration of the coupled Einstein-Proca system is described by the three functions $f = f(r), g = g(r)$, and $u = u(r)$ which enter the spherically symmetric ansatz for the Proca field as follows

$$A = \frac{u}{rf} \vartheta^{\hat{0}} = \frac{u}{r} d\tau. \tag{3.4}$$

Substitution of (3.1)-(3.4) into the Proca field equation (2.2) results in

$$\left\{ \frac{1}{r^2} \frac{f}{g} \left[\frac{r^2}{g} \left(\frac{u}{r} \right)' \right]' - \frac{m^2 u}{rf} \right\} \eta_{\hat{0}} = 0, \tag{3.5}$$

or, equivalently,

$$u'' - \frac{g'}{g} \left(u' - \frac{u}{r} \right) - \frac{m^2 g^2}{f^2} u = 0. \tag{3.6}$$

A direct calculation of the energy-momentum 3-form yields

$$\Sigma_\alpha = \frac{1}{2r^2 g^2 f^2} \left\{ -f^2 \left(u' - \frac{u}{r} \right)^2 + m^2 g^2 u^2 \right\} \delta_\alpha^{\hat{1}} \eta_{\hat{1}}$$
$$+ \frac{1}{2r^2 g^2 f^2} \left\{ f^2 \left(u' - \frac{u}{r} \right)^2 + m^2 g^2 u^2 \right\} \left(-\delta_\alpha^{\hat{0}} \eta_{\hat{0}} + \delta_\alpha^{\hat{2}} \eta_{\hat{2}} + \delta_\alpha^{\hat{3}} \eta_{\hat{3}} \right). \tag{3.7}$$

For the sake of completeness we also write down the Einstein 3-form

$$\frac{1}{2} R^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma} = \frac{f^2}{r^2 g^2} \left\{ \left[2r \frac{f'}{f} + 1 - \frac{g^2}{f^2} \right] \delta_\alpha^1 \eta_1 + \left[2r \left(\frac{f'}{f} - \frac{g'}{g} \right) + 1 - \frac{g^2}{f^2} \right] \delta_\alpha^{\hat{0}} \eta_{\hat{0}} \right\}$$
$$+ \frac{f^2}{r^2 g^2} \left\{ r^2 \frac{f''}{f} + \left(r \frac{f'}{f} \right)^2 - r^2 \frac{f'}{f} \frac{g'}{g} + 2r \frac{f'}{f} - r \frac{g'}{g} \right\} \left(\delta_\alpha^{\hat{2}} \eta_{\hat{2}} + \delta_\alpha^{\hat{3}} \eta_{\hat{3}} \right). \tag{3.8}$$

Inserting (3.7) and (3.8) into (2.3) we find (after some algebra) that the equations corresponding to $\alpha = \hat{0}, \hat{1}$ are

$$2 \left(r (f^2)' + f^2 - g^2 \right) = \kappa m^2 \frac{g^2}{f^2} u^2 - \kappa \left(u' - \frac{u}{r} \right)^2, \tag{3.9}$$

$$r (g^2)' = \kappa m^2 \frac{g^4}{f^4} u^2. \tag{3.10}$$

Furthermore, it is straightforward to see that the (second order) equation corresponding to $\alpha = \hat{2}, \hat{3}$ is a consequence of (3.9)-(3.10) and (3.6).

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The scalar invariant $|\Sigma| := {}^*(\Sigma_\alpha \wedge {}^*\Sigma^\alpha)$ characterizes the "magnitude" of the energy-momentum of the massive vector field. Using (3.7), we find that

$$|\Sigma| = {}^*(\Sigma_\alpha \wedge {}^*\Sigma^\alpha) = \frac{1}{r^4 g^4} \left\{ \left(u' - \frac{u}{r} \right)^4 + m^2 u^2 \frac{g^2}{f^2} \left(u' - \frac{u}{r} \right)^2 + m^4 u^4 \frac{g^4}{f^4} \right\}. \tag{3.11}$$

IV. PRELIMINARY ANALYSIS

Before we start the study of the complete system it is instructive to recall two particular cases: namely, massless vector particle in curved spacetime and massive vector particle in Minkowski spacetime.

For the *massless* vector particle

$$m = 0, \tag{4.1}$$

and one finds, from (3.10), that $g = g_0 = const$. Consequently, the vector field equation can now be easily integrated to give the usual Coulomb solution

$$u = q = const \implies A = \frac{q}{r} d\tau. \tag{4.2}$$

[Strictly speaking, the general solution reads $u = q + \beta r$, but one can put $\beta = 0$ since it contributes only an exact form to A]. Turning to (3.9), we immediately recover the well known Reissner-Nordström solution

$$f^2 = g_0^2 \left(1 - \frac{2M}{r} + \kappa \frac{q^2}{2g_0^2 r^2} \right), \tag{4.3}$$

with the integration constant M interpreted as the mass of the gravitating source and q/g_0 as its electric charge.

On the other hand, for the Minkowski spacetime the metric functions are $f = g = 1$ and the solution of the Proca field equation (3.6) yields the well known Yukawa potential

$$u = q e^{-mr} \implies A = \frac{q e^{-mr}}{r} d\tau. \tag{4.4}$$

This shows that the vector field is practically zero at distances much greater than the typical length $r_0 = 1/m$.

We expect that a spherically symmetric configuration of coupled Einstein and Proca fields will combine both of the typical features of the above limiting cases. Namely, for small values of mass m there will be a large part of space inside the sphere of radius $r_0 = 1/m$ where the function u is to a high degree of approximation constant. In this region the exact solution will naturally be approximated by (4.2) and the metric will assume the familiar Reissner-Nordström form (4.3). However, due to its massiveness the field A will remain, also in curved spacetime, confined to a finite spatial volume, whereas for $r \to \infty$ one expects a fast decay $u \to 0$ which leaves one with pure Schwarzschild metric.

The following observation will be very useful in the discussion of exact solutions. Multiplying (3.6) by u/g and using the Leibniz rule one finds that

$$\left[\frac{u}{g} \left(u' - \frac{u}{r} \right) \right]' = \frac{1}{g} \left\{ \left(u' - \frac{u}{r} \right)^2 + m^2 \, u^2 \, \frac{g^2}{f^2} \right\}, \tag{4.5}$$

where the right-hand side is positive definite in a regular spacetime region.

Identity (4.5) represents a particular case of the general relation

$$db = -(dA \wedge {}^*dA + m^2 A \wedge {}^*A), \tag{4.6}$$

where $b := -A \wedge {}^*dA$. The latter identity holds true for all solutions of the Proca field equation (2.2).

V. DIMENSIONLESS SYSTEMS

The general system is nonlinear and apparently cannot be integrated analytically. Consequently, we will present the results of numerical integration in the remainder of this paper. Before we begin the numerical analysis, we introduce a new *dimensionless* radial variable

$$\rho := \frac{r}{\sqrt{\kappa}}, \tag{5.1}$$

which allows us to rewrite the system (3.9), (3.10), and (3.6) in the form

$$2\left(\rho\frac{dF}{d\rho}+F-G\right)=K\frac{G}{F}u^2-\left(\frac{du}{d\rho}-\frac{u}{\rho}\right)^2,\tag{5.2}$$

$$\rho\frac{dG}{d\rho}=K\frac{G^2}{F^2}u^2,\tag{5.3}$$

$$\frac{d^2u}{d\rho^2}=K\frac{G}{F}u+\frac{1}{2G}\frac{dG}{d\rho}\left(\frac{du}{d\rho}-\frac{u}{\rho}\right).\tag{5.4}$$

Here we have introduced a dimensionless constant

$$K:=\kappa\,m^2,\tag{5.5}$$

and defined the functions

$$F:=f^2,\qquad G:=g^2.\tag{5.6}$$

Evidently, the dimensionless radial coordinate measures distance from the origin in units of the Planck length ($\kappa=\ell^2$). At the same time, the parameter $\sqrt{K}$, being the ratio of the Planck length to the Compton length of the vector particle, characterizes the size of the domain where the influence of the Proca field on the spacetime geometry is significant.

One can, alternatively, study a different dimensionless system after defining the scaled metric functions

$$\widetilde{F}:=F/K,\qquad \widetilde{G}:=G/K,\tag{5.7}$$

and introducing a new radial coordinate

$$\xi:=\sqrt{K}\rho=m\,r=\frac{r}{r_0}.\tag{5.8}$$

The system (5.2)-(5.4) then reads

$$2\left(\xi\frac{d\widetilde{F}}{d\xi}+\widetilde{F}-\widetilde{G}\right)=\frac{\widetilde{G}}{\widetilde{F}}u^2-\left(\frac{du}{d\xi}-\frac{u}{\xi}\right)^2,\tag{5.9}$$

$$\xi\frac{d\widetilde{G}}{d\xi}=\frac{\widetilde{G}^2}{\widetilde{F}^2}u^2,\tag{5.10}$$

$$\frac{d^2u}{d\xi^2}=\frac{\widetilde{G}}{\widetilde{F}}u+\frac{1}{2\widetilde{G}}\frac{d\widetilde{G}}{d\xi}\left(\frac{du}{d\xi}-\frac{u}{\xi}\right).\tag{5.11}$$

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In this form the equations no longer contain a free parameter (such as K) and the new dimensionless coordinate measures distance in units of the characteristic ("Compton wavelength") scale r_0. It is convenient to use both dimensionless systems. The advantage of (5.9)-(5.11) lies in the absence of K, whereas the equations (5.2)-(5.4) are more transparent from the physical point of view when one considers limits of small and big mass m.

VI. CONDITIONS AT THE ORIGIN AND AT INFINITY

Before one can start the numerical integration, an appropriate set of initial conditions must be specified. Unfortunately, the solution cannot be represented by analytic power series expansion for u, F, G at the origin in view of the apparent singularity at $\rho = 0$.

Instead, one can verify that for small values of ρ, irrespective of the value of K, there is an approximate solution of the form

$$u \approx q + b\,\rho, \tag{6.1}$$

$$F \approx \frac{1}{2}\frac{q^2}{\rho^2}, \tag{6.2}$$

$$G \approx \frac{1}{\frac{1}{c} - \frac{K}{q^2}\rho^4}, \tag{6.3}$$

where q, b, c are parameters which determine the initial conditions in the neighborhood of the origin $\rho = 0$.

At infinity, $\rho \to \infty$, following the physical discussion in Sect. IV, we expect approximate behavior of the form

$$u \longrightarrow u_0 \, \exp(-\sqrt{K}\rho), \tag{6.4}$$

$$F \longrightarrow g_0^2 \left(1 - \frac{2M}{\rho}\right), \tag{6.5}$$

$$G \longrightarrow g_0^2, \tag{6.6}$$

where u_0, g_0 are constants. The condition (6.4) means that the massive vector field is nontrivial only inside a sphere of a finite radius $1/\sqrt{K}$ ("Yukawa-type" behavior). On the other hand, the conditions (6.5)-(6.6) specify purely Schwarzschild asymptotic metric. A

more precise form of the limit (6.6) is easily obtained after substituting (6.4) and (6.5) into (5.2)-(5.4):

$$G \approx g_0^2 \left(1 + K\, u_0^2 \operatorname{Ei}(-2\sqrt{K}\rho) \right),$$ (6.7)

where $\operatorname{Ei}(x) = \int\limits_{-\infty}^{x} \frac{e^t}{t} dt$ is the integral exponential function. It is worthwhile to recall that asymptotically, for $x \to \infty$, one has $\operatorname{Ei}(-x) \approx -e^{-x}/x$.

We will use the asymptotic conditions (6.4)-(6.6), (6.7) in the numerical analysis of the problem under consideration.

VII. ABSENCE OF SOLUTIONS WITH HORIZONS

In this section we show that the spherically symmetric Einstein-Proca system does not admit asymptotically flat solutions with horizons. The absence of black holes for a massive vector field was first demonstrated by Bekenstein [8].

Let us consider an arbitrary regular solution $u(r)$ which vanishes at *two* points r_1 and $r_2 > r_1$: $u(r_1) = u(r_2) = 0$. Then $u(r) = 0$ for all $r_1 \le r \le r_2$. Indeed, integrating the identity (4.5) from r_1 to r_2, one finds that

$$\int\limits_{r_1}^{r_2} \frac{1}{g} \left\{ \left(u' - \frac{u}{r} \right)^2 + m^2\, u^2\, \frac{g^2}{f^2} \right\} dr = \frac{u(r_2)}{g(r_2)} \left(u'(r_2) - \frac{u(r_2)}{r_2} \right) - \frac{u(r_1)}{g(r_1)} \left(u'(r_1) - \frac{u(r_1)}{r_1} \right) = 0.$$ (7.1)

Since the integrand is positive definite, the vanishing of the integral leads to the above conclusion.

Consequently, a nontrivial solution $u(r)$ which vanishes *asymptotically* at $r_2 = \infty$ (thus satisfying the condition (6.4)) cannot have zeros at any finite r_1 (since then the solution would be trivial: $u(r) = 0$ for $r \ge r_1$).

This leads to the absence of the black hole type solutions of the system (5.2)-(5.4). In order to see this, let us recall that a black hole necessarily possesses a horizon. Quite generally, on a spacetime manifold $\mathcal{M}$ a horizon is defined as a hypersurface $S := \{x^i \in \mathcal{M} \mid \sigma(x^i) = 0\}$ such that: (i) the normal vector $n_i := \partial_i \sigma$ is null

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$$n_i \, n^i|_S = 0, \tag{7.2}$$

and (ii) S is not an essential singularity. The latter means that all the curvature invariants as well as the volume 4–form η are nonsingular on the horizon. In particular, the regularity of η follows from the condition (3.3) on the function g.

For a spherically symmetric gravitational field configuration, horizon S is evidently a sphere $\sigma = r = r_h$. Normal vector is then $n_i = \delta_i^1$. Substituting (3.1) into (7.2), one obtains

$$\frac{f^2(r_h)}{g^2(r_h)} = \frac{F(r_h)}{G(r_h)} = 0. \tag{7.3}$$

Since $G(r_h)$ is finite in view of (3.3), we find that F must vanish on the horizon S

$$F(r_h) = 0. \tag{7.4}$$

The last equation formally defines the position of a horizon in the general spherically symmetric spacetime (3.1). Now recall the second requirement: a hypersurface S must be free of physical singularities in order to be a horizon. Clearly, the energy-momentum invariant scalar (3.11) is regular at $r = r_h$ if and only if

$$u(r_h) = 0. \tag{7.5}$$

Furthermore, if (7.5) did not hold then the quadratic curvature invariants (obtained by using the Einstein field equations (2.3) in the definition of $|\Sigma|$) would diverge at r_h because of the last term in (3.11) and (7.4).

Now we are in a position to conclude that there are no solutions with a horizon and a nontrivial massive vector field. Indeed, assume the contrary is true. Then outside a horizon S the function u is necessarily given by $u(r) = 0$, $r_h \le r \le \infty$, because u vanishes at infinity (6.4) and at the horizon (7.5). Consequently, outside S, the system (5.9)-(5.10) has the usual Schwarzschild solution $G = 1, F = 1 - r_h/r$. Integrating (5.9)-(5.11) from r_h to 0 with the initial conditions (7.4) and (7.5), we find $u(r) = 0$ everywhere.

Bekenstein's original proof [8] was based on the assumption that the three-form $b = b^\alpha \eta_\alpha$ defined in (4.6) is bounded on the horizon. It is easy to see that $b^\alpha = \frac{u}{r^2 fg}\left(u' - \frac{u}{r}\right)\delta_1^\alpha$

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diverges on the horizon (7.4) unless u vanishes. Thus, for a massive vector field, the form b is not only bounded but, in fact, trivial on the horizon.

VIII. NUMERICAL SOLUTIONS

After fixing the value of the parameter K to the square of the ratio of the Planck length to the Compton wavelength of the vector field, one can start numerical integration at an arbitrarily small ρ with the initial conditions defined by (6.1)-(6.3). One is free to choose any initial value for the "boson charge" function $u(0) = q$ ($\neq 0$, otherwise u is trivial everywhere). Solutions with the correct asymptotic behavior (6.4)-(6.6) exist only for fixed values of the parameters $b = u'(0), c = G(0)$. Technically, the numerical integration can start at a point arbitrarily close to the origin for every chosen values of K and q. In order to obtain the asymptotic behavior (6.4)-(6.6), a fine tuning of b and c is required which can be achieved similarly to the construction of the Bartnik-McKinnon solutions [26] or of the Abrikosov-Nielsen-Olesen vortices (see, e.g. [27] and references therein). Alternatively, one can start the numerical integration at a sufficiently large radius with the initial conditions taken from (6.4)-(6.6) for arbitrary values of K, M, u_0. As a cross-check, we have used both integration schemes. The resulting approximate solutions turned out to be completely consistent with each other.

Particular solutions for various values of K, q and M are described in Tables I-III. The graphical form of the solutions is presented in Figures 1,2, and 3. In these figures, the numerical solutions are depicted for $K = 1$ and $M = 0.1$ (dotted lines), $M = 0.5$ and $M = 1.5$ (broken lines), and $M = 2$ (solid lines). In all cases we put $g_0 = 1$ which is always possible to achieve by the redefinition of the time coordinate $\tau \to g_0\tau$.

As one can see, the relation between K (formal rest mass of the vector field) and M (asymptotic total mass of the solution) plays a decisive role. At the same time, the value of the boson charge q at the origin is also important.

In agreement with the results of Bekenstein et al [8–10] and with the preliminary analysis

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of Sect. VII, all the numerical solutions obtained by us are without horizons. They possess a true physical singularity at the origin which provides us with an example of a *naked* spacetime singularity. Stability of these solutions against small perturbations will be studied separately.

Recalling that the effective Proca field emerges naturally in the general metric-affine models, we thus conclude that the presence of the post-Riemannian geometric objects prevents, in general, a formation of a black hole in MAG theory. Only in the special case when the MAG coupling constants are such that the effective mass vanishes, $m^2 = 0$, the black holes can be formed [20,22].

REFERENCES

[1] W. Pauli, *Rev. Mod. Phys.* **13** (1941) 203; E.M. Corson, *Introduction to tensors, spinors, and relativistic wave equations* (Blackie: London, 1953); H. Umezawa, *Quantum field theory* (North Holland: Amsterdam, 1956).

[2] A.S. Goldhaber and M.M. Nieto, *Rev. Mod. Phys.* **43** (1971) 277.

[3] E. Comay, *Nuovo Cim.* **113B** (1998) 733; P. Hillion and S. Quinnez, *Int. J. Theor. Phys.* **25** (1986) 727.

[4] N. Banerjee and R. Banerjee, *Mod. Phys. Lett.* **A11** (1996) 1919; J. Camacaro, R. Gaitan, and L. Leal, *Mod. Phys. Lett.* **A12** (1997) 3081; V. Aldaya, M. Calixto, and M. Navarro, *Int. J. Mod. Phys.* **A12** (1997) 3609; Y.-W. Kim, M.-I. Park, Y.-J. Park, and S.J. Yoon, *Int. J. Mod. Phys.* **A12** (1997) 4217; A.S. Vytheeswaran, *Int. J. Mod. Phys.* **A13** (1998) 765; M.-I. Park, and Y.-J. Park, *Int. J. Mod. Phys.* **A13** (1998) 2179.

[5] G. Velo and D. Zwanziger, *Phys. Rev.* **188** (1969) 2218.

[6] Yu.N. Obukhov, *J. Phys.* **A16** (1983) 3795; Yu.N. Obukhov, in: *Contrib. Papers of 10th Int. Conf. "General Relativity and Gravitation" (Padova, 4-9 July 1983)*, Eds. B. Bertotti, F. De Felice, and A. Pascolini (Cons. Naz. delle Ric.: Rome, 1983), vol. 1, p. 599-601.

[7] M. Seitz, *Class. Quantum Grav.* **3** (1986) 1265; R. Spinosa, *Class. Quantum Grav.* **4** (1987) 473; V.G. Bagrov, A.Yu. Trifonov, and A.A. Evseevich, *Class. Quantum Grav.* **9** (1992) 533.

[8] J. Bekenstein, *Phys. Rev. Lett.* **28** (1972) 452; J. Bekenstein, *Phys. Rev.* **D5** (1972) 1239; J. Bekenstein, *Phys. Rev.* **D5** (1972) 2403.

[9] G.V. Isaev, *Massive vector field and horizon surface*, in: *"Problems of gravity and elementary particle theory" / Ed. K.P. Stanyukovich* (Atomizdat: Moscow, 1976) v. **7**, p. 138-141 (in Russian).

[10] S.L. Adler and R.B. Pearson, *Phys. Rev.* **D18** (1978) 2798.

[11] V.P. Frolov, *Gen. Relat. Grav.* **9** (1978) 569.

[12] D. Gottlieb, R. Hojman, L.H. Rodrigues, and N. Zamorano, *Nuovo Cim.* **B80** (1984) 62.

[13] B. Leaute and B. Linet, *Gen. Rel. Grav.* **17** (1985) 783.

[14] B.R. Greene, S.D. Mathur, and C.M. O'Neill, *Phys. Rev.* **D47** (1993) 2242; M.E. Ortiz, *Phys. Rev.* **D45** (1992) R2586; K. Lee, V.P. Nair, and E.J. Weinberg, *Phys. Rev.* **D45** (1992) 2751.

[15] N. Rosen, *Found. Phys.* **24** (1994) 1689.

[16] R. Ruffini and S. Bonazzola, *Phys. Rev.* **187** (1969) 1767.

[17] E.W. Mielke and F.E. Schunck, *Boson Stars: Early history and recent prospects*, in: *Proc. of 8th Marcel-Grossmann Meeting on "Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field Theories", Jerusalem, Israel, 22-27 Jun 1997*, Ed. T. Piran (World Scientific, Singapore 1999, to appear); e-Print Archive: gr-qc/9801063.

[18] F.W. Hehl, J.D. McCrea, E.W. Mielke, and Y. Ne'eman, *Phys. Rep.* **258** (1995) 1.

[19] V.N. Ponomariov and Yu. Obukhov, *Gen. Relat. Grav.* **14** (1982) 309.

[20] R.W. Tucker and C. Wang, *Class. Quantum Grav.* **12** (1995) 2587.

[21] H.A. Buchdahl, *J. Phys. A: Math. and Gen.* **A12** (1979) 1235.

[22] Yu.N. Obukhov, E.J. Vlachynsky, W. Esser, R. Tresguerres, and F.W. Hehl, *Phys. Lett.* **A220** (1996) 1; E.J. Vlachynsky, R. Tresguerres, Yu.N. Obukhov, and F.W. Hehl, *Class. Quantum Grav.* **13** (1996) 3253.

[23] T. Dereli, M. Önder, J. Schray, R.W. Tucker, and C. Wang, *Class. Quantum Grav.*

13 (1996) L103; R.W. Tucker and C. Wang, in: *"Mathematics of gravitation. Part II. Gravitational wave detection"*, *Proc. of the Workshop "Mathematical Aspects of Theories of Gravitation"*, *(Warsaw, Poland, 26 Feb - 30 Mar 1996)*, Banach Center Publications, Vol. **41** (Inst. of Math., Polish Acad. Sci.: Warszawa, 1997) 263.

[24] Yu.N. Obukhov, E.J. Vlachynsky, W. Esser and F.W. Hehl, *Phys. Rev.* **D56** (1997) 7769.

[25] F.W. Hehl and A. Macías, *Metric affine gauge theory of gravity. 2. Exact solutions*, *Preprint UAM-I-9902026 (Mexico City Univ., Iztapalapa: February 1999)* 25pp; e-Print Archive: gr-qc/9902076; to be publ. in *Int. J. Mod. Phys.* **A** (1999).

[26] R. Bartnik and J. McKinnon, *Phys. Rev. Lett.* **61** (1988) 141.

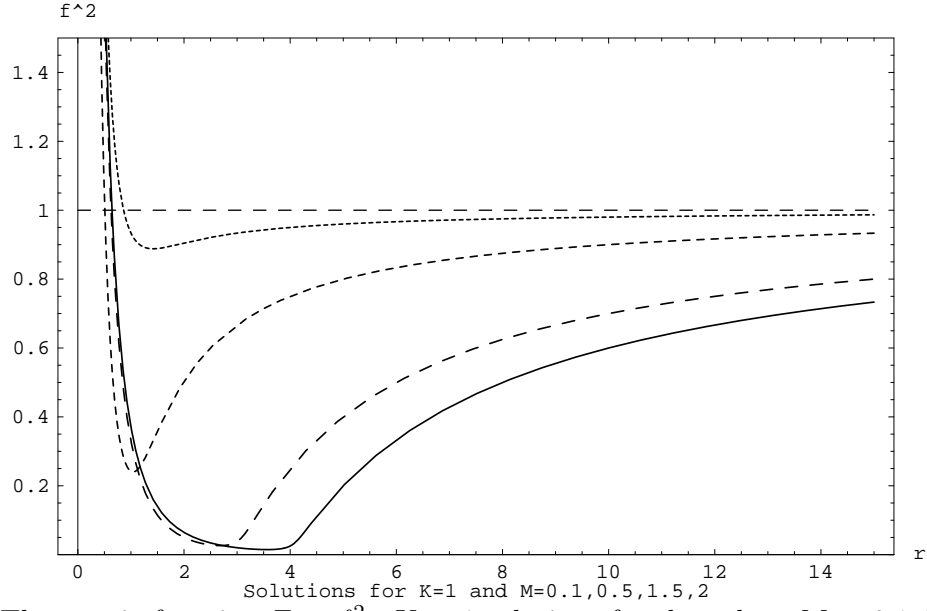[27] Yu.N. Obukhov and F.S. Schunck, *Phys. Rev.* **D55** (1997) 2307.

FIG. 1. The metric function $F = f^2$: $K = 1$ solutions for the values $M = 0.1$ (dotted line), $M = 0.5$ and $M = 1.5$ (broken lines), and $M = 2$ (solid line).
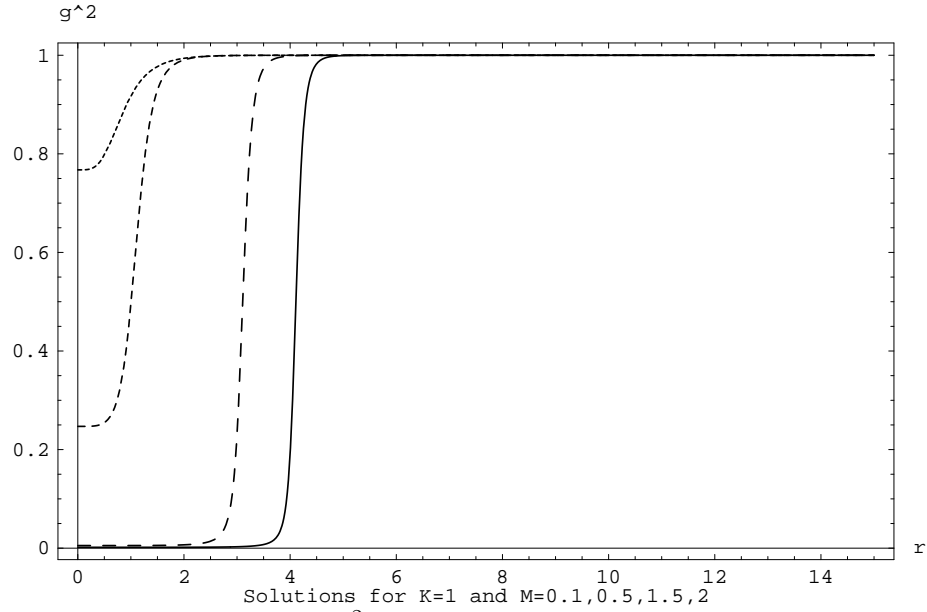

FIG. 2. The metric function $G = g^2$: $K = 1$ solutions for the values $M = 0.1$ (dotted line), $M = 0.5$ and $M = 1.5$ (broken lines), and $M = 2$ (solid line).
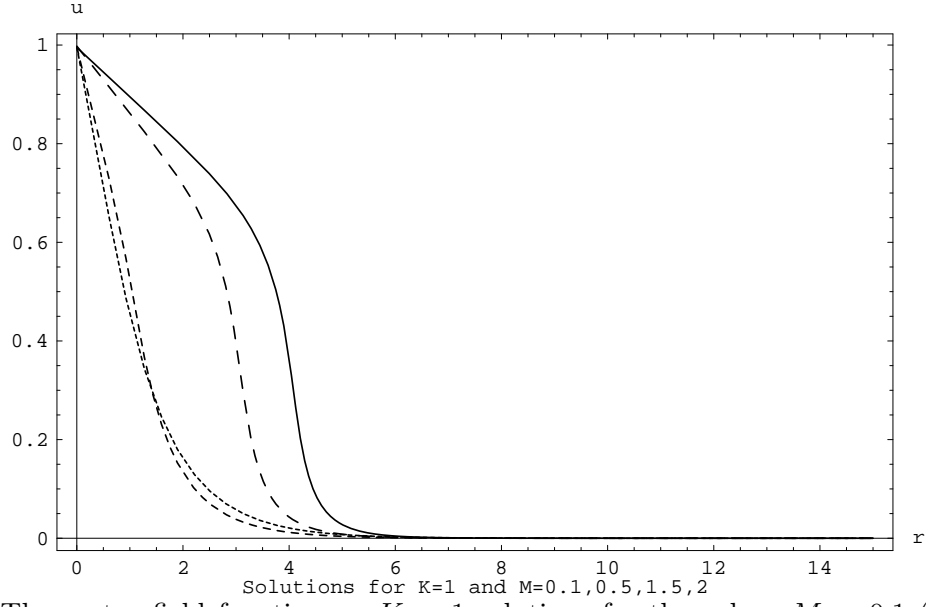
FIG. 3. The vector field function u: $K = 1$ solutions for the values $M = 0.1$ (dotted line), $M = 0.5$ and $M = 1.5$ (broken lines), and $M = 2$ (solid line).

TABLES

TABLE I. Solutions with fixed values of q and M

	$q = 4$ and $M = 0.5$			$q = 2$ and $M = 1$	
K	b	c	K	b	c
0.01	-0.29605	0.90444	10^{-6}	-0.00202	0.99996
0.10	-0.66479	0.69257	0.01	-0.16267	0.84124
1.00	-1.20050	0.34457	1.00	-0.56351	0.08738
10.00	-1.67696	0.08873	10.00	-0.69666	0.01100

TABLE II. $K = 1.00$: solutions for $q = 1$ with different masses M

M	b	c
0.10	-0.56959	0.76748
0.50	-0.43469	0.24706
1.50	-0.13588	0.00528
2.00	-0.10007	0.00170

TABLE III. Solutions for $M = 1$ and different q

	$q = 0.50$		$q = 1.00$	
K	b	c	b	c
0.01	0.60874	0.08438	0.25196	0.34048
1.00	-0.01728	0.00579	-0.21296	0.02528
10.00	-0.13175	0.00071	-0.33056	0.00308